An Introduction to Fiber Bundles and Fibrations

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Fiber bundles and fibrations play a central role in the theory of tautological rings and characteristic classes. They generalize the familiar notion of a covering space in homotopy theory, and also relate to the notion of a sheaf in algebraic geometry. We will give a brief introduction to fiber bundles and fibrations. We direct the reader who wishes to find a more thorough introduction to the material to read from [Hat09] or [Hus75]. We will begin via a specific (and very important) example of a fiber bundle called a vector bundle.

Definition 0.1 (Vector bundle)**.** *Let $\pi : E \to B$ be a continuous surjection of topological spaces E and B. Then $\pi : E \to B$ is a k-dimensional real vector bundle if the following conditions are satisfied:*

- *For all $b \in B$, $\pi^{-1}(b)$ is a finite-dimensional real vector space of dimension k.*
- *There exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of B such that for all U_α there exist homeomorphisms*
$$\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$
taking $\pi^{-1}(b)$ to $\{b\} \times \mathbb{R}^n$ via a linear isomorphism.
- *If $\alpha, \beta \in I$, then the composition $\phi_\beta^{-1} \circ \phi_\alpha : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is well-defined and satisfies*
$$\phi_\beta^{-1} \circ \phi_\alpha(x, v) = (x, g_{\alpha\beta}(x)v)$$
for some $GL(k)$-valued function
$$g_{\alpha\beta} : U \cup V \to GL(k).$$

- *These maps satisfy*

$$g_{\alpha\alpha} = I \ \ \textit{and} \ \ g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = I$$

The maps ϕ_α are called the *local trivializations* of the vector bundle, the maps $\phi_\beta^{-1} \circ \phi_\alpha$ are called *transition functions*, the last condition is called the *cocycle condition*, and the spaces E and B are called, respectively, the total space and the base space. Thus if a map $\pi : E \to B$ has a vector bundle structure on it, we are saying that locally π looks like a projection map of the form $U \times \mathbb{R}^n \to U$. In other words, E is locally the product of B with $\mathbb{R}^n$. We would like to give the reader a few examples with which to understand this definition.

Example 0.2 (Trivial bundle)**.** *The most obvious example of a vector bundle is the natural projection map*

$$B \times \mathbb{R}^n \to B.$$

The local product structure exists because of the global product structure. This example is called the trivial bundle.

Example 0.3 (Möbius bundle)**.** *Consider the space $E = ([0,1] \times \mathbb{R})/\sim$, where $\sim$ is the equivalence relation $(0,x) \sim (1,-x)$. There is a retraction $E \to B = S^1$ via the map $(t,x) \mapsto t$, which clearly has the structure of a real vector bundle by taking any open cover of S^1 which does not contain the whole space.*

This is our first nontrivial example of a vector bundle. We can show this by using sections of the vector bundle. A section of a real vector bundle is simply a continuous map $s : B \to E$ such that $\pi \circ s = Id_B$. In other words, s is a section if it maps each point of B into its fiber and does so continuously.

Note that the trivial bundle has sections of the form $b \mapsto (b,t)$ for some constant t, in particular for which there is no $b \in B$ such that $b \mapsto (b,0)$ if $t \neq 0$. However, in the case of our bundle, it is obvious to see that every section must have a point $b \in S^1$ such that using the isomorphism $\phi : \pi^{-1}(b) \xrightarrow{\sim} \{b\} \times \mathbb{R}$, $\phi \circ s(b) = (b,0)$ because of the intermediate value theorem and the quotient by $(0,x) \sim (1,-x)$. Thus $\pi : E \to B$ is a nontrivial example of a real vector bundle.

Note that E is homeomorphic to the Möbius strip with its boundary circle deleted, and so we call this particular real vector bundle the Möbius bundle.

Example 0.4 (Tautological bundle)**.** *The final example of a real vector bundle which we will give is called the tautological bundle, which is a bundle over a Grassmanian* *manifold* $G_n(\mathbb{R}^{n+k})$. *Recall that the Grassmanian manifold, as a set, is the set of* n-*dimensional subspaces of* $\mathbb{R}^{n+k}$. *Thus we can construct a total space* E *as the set of all* (V, v) *where* $V \in G_n(\mathbb{R}^{n+k})$ *is an* n-*dimensional subspace of* $\mathbb{R}^{n+k}$ *and* $v \in V$. *We topologize this as a subspace of* $G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$ *and get a vector bundle structure via the obvious map* $\pi : (V, v) \mapsto V$.

Example 0.5 (Locally free sheaves)**.** *In the context of algebraic geometry and scheme theory, an example of a vector bundle is a locally free sheaf.*

The reader will note that, in all but Example 0.3, we did not use any specific properties of $\mathbb{R}$ beyond its vector space structure. There are analagous definitions and examples in the case that we replace $\mathbb{R}$ with $\mathbb{C}$, which is straightforward to work out. We leave it to the reader to work out the examples which correspond to Examples 0.2 and 0.5 in the complex case.

We can generalize these ideas further to the notion of a *fiber bundle*, in which we replace $\mathbb{R}$ or $\mathbb{C}$ from the examples above with any topological space X. Recall that for any map of topological spaces $f : A \to B$, the *fiber* of f over $a \in A$ is simply the preimage $f^{-1}(a)$.

Definition 0.6 (Fiber bundle)**.** *Let* $\pi : E \to B$ *be a continuous map of topological spaces* E *and* B. *Then* $\pi : E \to B$ *is a fiber bundle if the following conditions are satisfied:*

(i) *For all* $b \in B$, $\pi^{-1}(b)$ *is homeomorphic to a fixed topological space* F

(ii) *There is an open cover* $\{U_\alpha\}_{\alpha \in I}$ *with isomorphisms*

$$\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$$

which restricts on fibers to a homeomorphism.

(iii) *If* $\alpha, \beta \in I$, *then the composition* $\phi_\beta^{-1} \circ \phi_\alpha : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ *is well-defined and satisfies*

$$\phi_\beta^{-1} \circ \phi_\alpha(x, v) = (x, g_{\alpha\beta}(x)v)$$

for some $Aut(F)$-*valued function*

$$g_{\alpha\beta} : U \cup V \to Aut(F).$$

(iv) These maps satisfy

$$g_{\alpha\alpha} = Id \text{ and } g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = Id$$

As before, the $\phi_\beta^{-1} \circ \phi_\alpha$ are called transition functions, the last condition is called the cocycle condition, E is called the total space, B is the base space and we call F the *fiber*. Diagrammatically, a fiber bundle is often drawn as

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d] \\ & B, \end{tikzcd}$$

which gives it the feel of a "short exact sequence of spaces." One can think of fiber bundles intuitively as a quotient, similar to a group quotient, where F takes on the same role that a normal subgroup plays in a group quotient.

As we might hope with any object we define, fiber bundles form a category. Maps of fiber bundles, which we simply call *bundle maps*, are commuting squares

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ B \arrow[r] & B', \end{tikzcd} \tag{1}$$

where $E \to B$ and $E' \to B'$ are fiber bundles and all the maps are continuous. If we fix a base space B, we can define a category of fiber bundles over B by defining morphisms to be commuting squares such as (1) with the condition that the map on the bottom row be the identity map.

One important feature of fiber bundles is that they have the homotopy lifting property. Recall that $\pi : E \to B$ has the homotopy lifting property with respect to a space X if, for all homotopies

$$h : [0, 1] \times X \to B,$$

if there exists a map $f_0 : \{0\} \times X \to E$ such that $\pi \circ f_0 = h|_{\{0\}\times X}$, then there exists a homotopy $f : [0, 1] \times X \to E$ such that $\pi \circ f = h$ and $f|_{\{0\}\times X} = f_0$. A *fibration* is a surjection $\pi : E \to B$ which satisfies the homotopy lifting property with respect to any space.

Example 0.7 (Covering spaces)**.** *If F is discrete we get a fiber bundle which is familiar to any beginning student of algebraic topology, namely a covering space.*

Example 0.8 (Hopf fibration). *One famous example of a fibration which is applicable to homotopy theory is the Hopf fibration*

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^3 \arrow[d, "p"] \\ & S^2. \end{tikzcd}$$

The map $p : S^3 \to S^2$ can be constructed by giving S^3 the structure of the complex subspace of $\mathbb{C}^2$ given by $\{(z_0, z_1) \,|\, |z_0|^2 + |z_1|^2 = 1\}$ and S^2 the structure of a subspace of $\mathbb{C} \times \mathbb{R}$ given by $\{(z, x) \,|\, |z|^2 + x^2 = 1\}$. Then p is given by

$$(z_0, z_1) \mapsto (2z_0 z_1^*, |z_0|^2 - |z_1|^2).$$

As denoted by the name, it is a standard result that $p : S^3 \to S^2$ is a fibration.

Example 0.9 (Homotopy Fibration). *In this example we will give an important construction of a fibration, called the homotopy fibration, which we use in Section* **??**.

Given any map $f : E \to B$, we can associate a topological space

$$E_f := \{(e, p) \,|\, e \in E \text{ and } p : I \to B \text{ such that } p(0) = f(e)\}.$$

E_f is topologized as a subspace of $E \times B^I$, where B^I is the function space of paths in B. There is a natural map, given by

$$E_f \xrightarrow{f'} B : (e, p) \mapsto p(1),$$

which we claim is a fibration. To show this, consider a homotopy $g : I \times X \to B$ and take a map $\tilde{g}_0 : X \to E_f$ such that $\tilde{g}_0 \circ f' = g_0$, where $g_0 = g|_{\{0\}\times X}$. We can extend $\tilde{g}_0$ to a homotopy $\tilde{g} : I \times X \to E_f$ which lifts g in the following way: Let γ_x be the image of $I \times \{x\}$ under g and we write $(e_x, \sigma_x) = \tilde{g}_0(x)$. We define $\tilde{g}$ to be the map $(t, x) \mapsto (e_x, \tilde{\gamma}_x(t))$, where $\tilde{\gamma}_x(t)$ is the path from $f(e_x)$ to $\gamma_x(t)$ which follows the path $\gamma_x \circ \sigma_x$. It is easy to see that $\tilde{g}$ lifts g, and so $f' : E_f \to B$ is a fibration.

Note that we can embed E into E_f via the map $e \mapsto (e, p_{const(e)})$, where $p_{const(e)}$ is the constant map $I \to \{e\}$. By contracting the paths in E_f, we have that E_f deformation retracts onto E and thus that E and E_f are homotopy equivalent. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} E & \hookrightarrow & E_f \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f'} & \\ B. & & \end{array}$$

We call the fiber of a point $* \in B$ *under* f' *to be the homotopy fiber at* $*$. *One can think of* E_f *as a fattening of* E *which gives us desirable homotopy-theoretic properties, and the homotopy fiber simply as the fiber under this "fattening."*

There are variations of the definition of a fiber bundle, depending on the structure on the spaces which we care about. For example, if we care about a smooth structure on F, E and B then we alter the definition slightly to include that all the maps in question be smooth, and that the homeomorphisms be diffeomorphisms. Likewise, bundle maps have the additional condition that they be smooth maps. These are called *smooth bundles*. This leads us to an important example of a vector bundle.

Example 0.10 (Vertical tangent bundle)**.** *Let* M^n *be a smooth manifold embedded in some* $\mathbb{R}^N$, $N > n$. *The tangent bundle over* M *is the subset of* $M \times \mathbb{R}^n$ *defined by*

$$\{(m, v) \,|\, m \in M \subset \mathbb{R}^N \textit{ and } v \textit{ is in the tangent space of } m\}.$$

By this definition, every smooth manifold has a unique tangent bundle.

Given a smooth map of smooth manifolds, $f : X \to Y$, *recall the basic definition from calculus on manifolds that* Df *is a map from the tangent bundle of* X *to the tangent bundle of* Y *which restricts to a linear map on fibers. If* Df *is a surjection on the tangent space of each point, there exists a natural vector bundle which is associated to* f, *called the vertical tangent bundle, which is defined simply by* $T_f := \ker Df$.

Another variation, which we will introduce in the next section, is called a principal G-bundle. We mention one final result which illustrates the importance of fibrations in the study of homotopy groups.

Theorem 0.11 (Theorem 4.41 of [Hat05])**.** *Let* $\pi : E \to B$ *be a fibration with* B *path connected. Then there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

Because the universal cover of S^1 is contractible, and a covering map is a fibration, it follows immediately that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_i(S^1) = 0$ for all $i \neq 1$. Then using, for example, the Hopf fibration $S^3 \to S^2$, the long exact sequence of homotopy groups gives that for all $i > 2$, $\pi_i(S^2) \cong \pi_i(S^3)$.

References

[Hat05] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2005.

[Hat09] Allen Hatcher. Vector bundles and K-theory. Version 2.1, 5 2009.

[Hus75] D. Husemoller. *Fibre Bundles*. Springer-Verlag Berlin and Heidelberg GmbH & Co. K, 1975.