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Classification of Vector Bundles Over $\mathbb{P}^1_k$

In this paper we will give a thorough proof of Grothendieck's classification theorem of vector bundles over $\mathbb{P}^1_k$, first given in [2]. Gorthendieck's original theorem was given in the language of representation theory, and applied to vector bundles on the Riemann sphere. What we will prove is essentially identical, except we will show the classification theorem for vector bundles over the scheme $\mathbb{P}^1_k$, over any field k, as outlined in [1], which was given in the prompt. Hereafter, any time we use the notation $\mathbb{P}^1$ we will be referring to $\mathbb{P}^1_k$.

Before giving a statement of the theorem that we'll prove, it will be useful to be specific about the conventions that we will use. Since terms such as 'vector bundle' mean slightly different things in different contexts, we will give precise definitions and constructions for all of the notions that we will use. We note here that most of the conventions and notation we will use will follow the conventions in [1]. This is not a summary of [1], but instead in some sense an enriching of it, where I provide in much greater detail the proof and concepts related to Grothendieck's theorem.

We will begin by outlining a standard construction of $\mathbb{P}^1_k$ via a glueing of two schemes as given in [4]. Let k be any field, and let

$$U_1 = \text{ Spec}(k[s]), U_2 = \text{ Spec}(k[t]),$$
$$U_{12} = \text{ Spec}(k[s,s^{-1}]) = U_1 \setminus \{0\}, U_{21} = \text{ Spec}(k[t,t^{-1}]) = U_2 \setminus \{0\}.$$

Then $\mathbb{P}^1_k$ is defined by glueing U_1 and U_2 together along U_{12} and U_{21} via the isomorphism

$$k[s,s^{-1}] \xrightarrow{\sim} k[t,t^{-1}]$$

given by $s \mapsto t^{-1}$. Note that $\mathbb{P}^1_k$ is a reduced, separated, finite type scheme over k, and is smooth, projective and geometrically integral. Note also that $\mathbb{P}^1_k$ is Noetherian because k is a field.

The reader might note here that this construction of $\mathbb{P}^1$ illustrates the fact that the *scheme* $\mathbb{P}^1_k$ is the same as the *variety* $\mathbb{P}^1_k$ because $k[t], k[s], k[t,t^{-1}]$, and $k[s,s^{-1}]$ are all PIDs, and thus every prime ideal is also maximal. In this document we will use scheme theoretic techniques, and regard $\mathbb{P}^1$ as a scheme in order to take advantage of more advanced machinery. In some cases, doing so might change the space we're interested in, as in general not every prime ideal is maximal in a commutative ring with unit. However, in the case of the projective line, considering it as a scheme changes nothing.

For the proofs of our two main theorems, we will need two different but equivalent notions of a *vector bundle*. Our first definition, used in the proof of Theorem 1.1, is the same one we saw in our lectures, which is:

Definition 0.1 (Vector Bundle)**.** *A vector bundle V of rank r over $\mathbb{P}^1_k$ is a coherent, locally free $\mathcal{O}_{\mathbb{P}^1_k}-$module such that over each open set U on which V is a free $\mathcal{O}_{\mathbb{P}^1_k}-$module, $V \cong \mathcal{O}_{\mathbb{P}^1_k}^{\oplus r}$. If $r = 1$, V is called a line bundle.*

The second, equivalent definition of a vector bundle comes from II.5.18 of [3], and is as follows:

Definition 0.2 (Vector Bundle). *Let X be a scheme. A vector bundle V of rank r over X consists of a scheme V, an affine morphism $f : V \to X$, an open cover $\{U_i\}$ of X and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{A}^n \times U_i$, such that for any i, j and any open affine subset $V = \text{Spec}(A) \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n \times V \cong \text{Spec}(A[x_1, ..., x_r])$ is given by a linear automorphism of $A[x_1, ..., x_n]$. Equivalently, ψ is given by a matrix in $GL(n, A)$.*

An isomorphism of vector bundles $(V, f, \{U_i\}, \{\psi_i\})$ and $(V', f'\{U_i'\}, \{\psi_i'\})$ of rank n over X is an isomorphism of schemes $g : V \xrightarrow{\sim} V'$ such that $f = f' \circ g$, and such that $(V, f, \{U_i\} \cup \{U_i'\}, \{\psi_i\} \cup \psi_i' \circ g)$ is a vector bundle over X.

We will use this definition in the proof of Proposition 2.1 later on. We make a short remark here that Definition 0.2 is in line with the standard definition of a vector bundle in topology and other areas; the additional structure that it requires makes it so that one can consider any vector bundle equivalently as a locally free, coherent $\mathcal{O}_X-$module, as in Definition 0.1.

We will now introduce the Picard group of $\mathbb{P}^1_k$, written $\text{Pic}(\mathbb{P}^1_k)$, which is the group of isomorphism classes of line bundles on $\mathbb{P}^1_k$, where the group operation is the tensor product. This group is generated by the tautological line bundle, $\mathcal{O}(1)$, or alternatively by its dual

$$\mathcal{O}(-1) := \mathcal{H}om(\mathcal{O}(1), \mathcal{O}_{\mathbb{P}^1_k}) = \mathcal{O}(1)^*.$$

Here $\mathcal{H}om(V, -)$ maps a sheaf F to the sheaf $U \mapsto \text{Hom}(V(U), F(U))$. We will write $\mathcal{O}(n)$ for $\mathcal{O}(1)^{\otimes n}$ and note that, in general,

$$\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n} = (\mathcal{O}(1)^*)^{\otimes n} = \mathcal{O}(n)^*,$$

as is given on page 50 of [5]. As it turns out, this group is freely generated by $\mathcal{O}(1)$, meaning that every line bundle over $\mathbb{P}^1_k$ can be written as $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$, and that $\mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n+m)$. Note that $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1_k}$ is the trivial element of $\text{Pic}(\mathbb{P}^1_k)$; it will be a corollary of our main theorem that $\text{Pic}(\mathbb{P}^1_k)$ is *freely* generated by $\mathcal{O}(1)$, and thus that $\text{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$. The *degree* of a line bundle $\mathcal{L}$ is the integer $n \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}(n)$.

Given a vector bundle V, we define $V(n) := V \otimes \mathcal{O}(n)$ and say that we *twist* V by $\mathcal{O}(n)$. This comes from an alternative name of $\mathcal{O}(1)$, which is the *Serre twisting sheaf.* There is an important and well-known result by Serre, which we will include here as well.

Theorem 0.1 (Serre's Vanishing Theorem). *For any ample line bundle L on a proper scheme X over a Noetherian ring, and any coherent sheaf F on X, there is an integer m_0 such that for all $m \geq m_0$, the sheaf $F \otimes L^{\otimes m}$ is spanned by its global sections and has no cohomology in positive degrees.*

Note that this result is simply one of the main properties of ample line bundles, and thus of $\mathcal{O}(1)$, which is ample.

Finally, we will define the sheaf ω_X for a topological space X as the sheaf of differentials on X. As it turns out, on $\mathbb{P}^1_k$, ω is simply the sheaf $\mathcal{O}(-2)$ (as stated in [1]). We make the concluding remark on notation, being that since H^i is the i^{th} right-derived functor of the global sections functor Γ, and Γ is left-exact, $H^0 \cong \Gamma$. Thus in this document, instead of referring to the global sections functor as Γ, we will simply use H^0.

1 Statement of the Theorem

Now that we have defined the basic notions which we need, we are ready to state the main theorem of our document.

Theorem 1.1 (Grothendieck's Classification of Vector Bundles Over $\mathbb{P}^1$)**.** *Let V be a vector bundle on $\mathbb{P}^1_k$. Then there exist integers $n_1, ..., n_q$ and $r_1, ..., r_q$ such that*

$$V = \mathcal{O}(n_1)^{r_1} \oplus \cdots \oplus \mathcal{O}(n_q)^{r_q}.$$

If we require that $n_1 > n_2 > \cdots > n_q$, this decomposition is unique.

As explained in [6], there are two main ways to prove this theorem. One involves a theorem by Dedekind and Weber, Theorem 1.3, and which is equivalent to Theorem 1.1. Theorem 1.3 is elementary in nature, in that it uses linear algebra and does not rely heavily on sheaf-theoretic techniques. The proof of equivalence is quite short and clear, and the bulk of proving this will be proving Theorem 1.3 itself. I am of the opinion that when possible and practical, it is better to give a proof which shows intuition and explicit construction. So, although proving our main result by proving Theorem 1.3 is both valid and fascinating, we will prove Theorem 1.1 by an alternative route.

The alternative way to prove this is much more scheme-theoretic and involves sheaf cohomology. It relies on the the Serre Duality Theorem, which we will state, but not prove.

Theorem 1.2 (Serre Duality)**.** *Let V be a vector bundle on X. Then there is an isomorphism of finite-dimensional $k-$vector spaces*

$$H^i(X, V) \xrightarrow{\sim} H^{1-i}(X, V^* \otimes \omega_X)$$

where V^* is the dual bundle of V given by $V^*(U) = Hom(V(U), \mathcal{O}_X(U))$, and ω_X is the sheaf of differentials on X.

In this document, then, we will prove the equivalence of Theorems 1.1 and 1.3, and then prove Theorem 1.1 using Serre Duality and sheaf cohomology. Both proofs which we supply will give the reader significant intuition for why this happens; indeed, in the proof of Theorem 1.1 we give an explicit construction of the decomposition into line bundles. We will now give the theorem of Dedekind and Weber.

Theorem 1.3 (Dedekind and Weber). *Let k be a field, x a variable, and consider $k[x,x^{-1}]$. Let $A \in GL(n,k[x,x^{-1}])$. Then there exist matrices $B \in GL(n,k[x])$, $C \in GL(n,k[x^{-1}])$ such that*

$$BAC = \begin{pmatrix} x^{d_1} & & 0 \\ & \ddots & \\ 0 & & x^{d_n} \end{pmatrix} \tag{1}$$

where $d_1 \geq d_2 \geq \cdots \geq d_n$ and the sequence of d_i's is unique.

2 Equivalence of Theorems 1.1 and 1.3

Just by looking at the main result of the Theorem 1.3, it seems intuitive that it might be somehow equivalent to Theorem 1.1. Indeed, if we can somehow represent our vector bundle V via an invertible matrix $A \in \mathrm{GL}(n,k[x,x^{-1}])$ then it is plausible that the notion of diagonalizability given in 1 will correspond to decomposing V into a sequence of line bundles, where the base change of multiplying A by B on the left and by C on the right would correspond to some isomorphism of vector bundles. Indeed, once these connections are established, the equivalence of theorems becomes merely a restatement of the same theorem with different language.

Proposition 2.1. *Theorems 1.1 and 1.3 are equivalent.*

Proof. Consider $\mathbb{P}^1$ as given in the introduction, and consider a vector bundle V over $\mathbb{P}^1$ of rank n. We will use the definition of a vector bundle given in Definition 0.2. If we consider the standard cover of $\mathbb{P}^1$, given by U_0 and U_1 as above, then V will be determined by the glueing map $\psi_0 \circ \psi_1^{-1}$, which, as given in the definition, is the automorphism $\mathrm{Spec}(k[x,x^{-1}])^n \xrightarrow{\sim} \mathrm{Spec}(k[x,x^{-1}])^n$ given by a linear automorphism $k[x,x^{-1}]^n \xrightarrow{\sim} k[x,x^{-1}]^n$. Equivalently, V is determined by a matrix $A \in \mathrm{GL}(n,k[x,x^{-1}])$.

Since A corresponds to the glueing, note that if we take $B \in \mathrm{GL}(n,k[x])$, BA just corresponds to the same vector bundle as A, except we've just changed coordinates on U_0. Likewise, taking $C \in \mathrm{GL}(n,k[x\ \in])$, the vector bundles given by AC and A are isomorphic. Indeed, in the case of $\mathbb{P}^1$ it is clear by the definition given that any isomorphism of vector bundles is simply given by some base change of basis of A of the form BAC, because any isomorphism of vector bundles is an isomorphism on the local trivial bundles after applying some automorphism to $\mathbb{P}^1$. So if $A \cong A'$ but $A \neq A'$, there must be some change of coordinates, given by an automorphism on $\mathbb{P}^1$ and thus a base change of the form BAC, such that $A' = BAC$.

Finally, by the construction of $\mathcal{O}(1)$ on page 43 of [5], $\mathcal{O}(n)$ corresponds to the isomorphism $k[x,x^{-1}] \xrightarrow{\sim} k[x,x^{-1}]$ given by $a \mapsto a^n$. Therefore, $\mathcal{O}(n)$ corresponds to the 1×1, invertible matrix $(x^n) \in \mathrm{GL}(1,k[x,x^{-1}])$.

With the clarifications just given, Theorems 1.1 and 1.3 are essentially the same statement. Explicitly, given a vector bundle V corresponding to $A \in \mathrm{GL}(n,k[x,x^{-1}])$, the isomorphism from Theorem 1.1 implies that there exist matrices $B \in \mathrm{GL}(n, k[x])$ and $C \in \mathrm{GL}(n, k[x^{-1}])$ such that

$$BAC = \begin{pmatrix} x^{d_1} & & 0 \\ & \ddots & \\ 0 & & x^{d_n} \end{pmatrix}$$

and $d_1 = \cdots = d_{r_1} = n_1, d_{r_1+1} = \cdots = d_{r_2} = n_2, ..., d_{r_{q-1}+1} = \cdots = d_{r_q} = n_q$. The uniqueness of the sequence $d_1 \geq \cdots \geq d_n$ comes from the uniqueness of the sequence of n_i and r_i from Theorem 1.1.

Conversely, Theorem 1.3 implies that V is isomorphic to a vector bundle of the form

$$\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n), \tag{2}$$

which corresponds to BAC from the statement of the theorem. We can rewrite (2) as

$$\mathcal{O}(n_1)^{r_1} \oplus \cdots \oplus \mathcal{O}(n_q)^{r_q}$$

such that $d_1 = \cdots = d_{r_1} = n_1$ and $d_{r_{i-1}+1} = \cdots = d_{r_i} = n_i$, for $1 < i \leq q$. Again, this decomposition is unique by uniqueness of the sequence $d_1 \leq \cdots \leq d_n$, and we have our result. $\square$

3 Proof of Grothendieck's Theorem

We will now move on to give the proof for our main result, Grothendieck's decomposition of vector bundles over $\mathbb{P}^1_k$. We will need a couple vital lemmas for our proof, supplied below. For the sake of continuity and flow of the document, we will not immediately prove the lemmas, but instead supply the proofs after the proof of Grothendieck's theorem.

Recall, as stated before, that we will be thinking of vector bundles in the sense of Definition 0.1, as coherent, locally free sheaves on $\mathbb{P}^1$.

Lemma 3.1. *Let $\varphi : W \to V$ be an injective map of vector bundles on $X = \mathbb{P}^1_k$. The quotient coherent module V/W may not be a vector bundle. There is, however, and extension W' of W in V such that V/W' is a vector bundle. The rank of W' (resp. V/W') is equal to that of W (resp. the generic rank of V/W). Here the generic rank is taken to mean the rank of the stalk at the generic point of X.*

Lemma 3.2. *Let V be a vector bundle on X. For all $i \geq 0$, we have the following isomorphism of functors:*

$$Ext^i(V, -) \cong H^i(V^* \otimes -) : Coh_{\mathcal{O}_X} \to Vec_k$$

Before proceeding to the full proof, we will give a brief outline of what's coming. Here, we consider a vector bundle V which, as shown in the introduction, is a coherent, (Zariski) locally free sheaf. Step 1: Find a maximal $n \in \mathbb{Z}$ such that twisting by $\mathcal{O}(-n)$ preserves the existence of nontrivial global sections, and twist. Step 2: Show that $H^0(V(-n)) \cong k^r$ for some r, and find an embedding of $\mathcal{O}^r$ into $V(-n)$ such that $V(-n)/\mathcal{O}^r$ is a vector bundle. Step 3: Show that the resulting exact sequence

$$0 \to \mathcal{O}^r \to V(-n) \to W(-n) \to 0$$

splits, giving that $V(-n) \cong \mathcal{O}^r \oplus W(-n)$. Step 4: Twist by $\mathcal{O}(n)$ and get

$$V \cong \mathcal{O}^r(n) \oplus W. \tag{3}$$

Step 5: Now consider $V' = W$ and repeat steps 1 through 4 on V', getting $V' = \mathcal{O}^{r_2}(-n_2) \oplus W'$. This process must terminate (as we will show). By setting $n_1 = n$, $r_1 = r$, for some q we have:

$$V = \mathcal{O}(n_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}(n_q)^{\oplus r_q},$$

as desired. Finally, Step 6 is to show that the decomposition in (3) is unique up to reordering of the direct summands.

Proof of Theorem 1.1. Let V be a nontrivial (i.e. nonzero) vector bundle on $\mathbb{P}^1_k$. We would like to find an integer $n \in \mathbb{Z}$ which is maximal with respect to the property that $V(-n)$ has nontrivial global sections, but not $V(-(n+1))$.

Note that if V has no global sections, then since V is coherent and locally free, and $\mathcal{O}(1)$ is ample, twisting by $\mathcal{O}(1)$ sufficiently many times makes V globally generated. Note that twisting by $\mathcal{O}(1)$, by definition, changes nothing locally on V because for small enough open sets U the tensor product is given by

$$V(U) \otimes_{\mathcal{O}} \mathcal{O}(U) \cong V(U).$$

Thus we know that for a sufficiently large m, $V(m) = V \otimes \mathcal{O}(m)$ is nontrivial and globally generated, and thus has global sections. If m' is the first integer for which $V(m')$ has nontrivial global sections (one such integer must exist because V has no global sections and $V(m)$ does), then setting $n = -m'$ gives our desired result.

On the other hand, if V has nontrivial global sections, then consider the identity given by Serre Duality:

$$H^0(\mathbb{P}^1, V(-n)) \cong H^1(\mathbb{P}^1, V^*(n) \otimes \omega) \cong H^1(\mathbb{P}^1, V^*(n-2)). \tag{4}$$

By Serre's Vanishing Theorem there exists some $m \in \mathbb{N}$, with $m \neq 0$, such that $H^1(\mathbb{P}^1, V^*(m-2)) \ncong 0$ but $H^1(\mathbb{P}^1, V^*((m+\ell)-2)) \cong 0$ for all $\ell > 0$. Thus letting $n = m$, we have that n is the maximal integer such that $H^0(V(-n)) \cong H^1(\mathbb{P}^1, V^*(m-2)) \ncong 0$ but $H^0(V(-(n+1))) \cong H^1(\mathbb{P}^1, V^*((m+1)-2)) \cong 0$, as desired.

Now that we have this n, recall that $V(-n)$ is a coherent module, and so in particular, the global sections $H^0(\mathbb{P}^1, V(-n))$ are a $k-$module because $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong k$ (Proposition 3.18 from the lecture notes, [5]). Furthermore, by coherence $H^0(\mathbb{P}^1, V(-n))$ is finitely generated, and so we have that $H^0(\mathbb{P}^1, V(-n))$ is a finite-dimensional $k-$vector space. So let $H^0(\mathbb{P}^1, V(-n)) \cong k^r$ for some $r \in \mathbb{N}$, and with the standard basis $e_1, ..., e_r$ and pick a global section $e_\ell \in k^r$. Taking $\mathcal{O}$ as a module over itself, we can use e_ℓ to define a map $\mathcal{O} \to V(-n)$ given by $1 \mapsto e_\ell$. This is clearly injective on global sections, and necessarily injective on sheaves.

A priori it does not necessarily follow that $\mathcal{O}$ is a sub-vector bundle of $V(-n)$, but if it's not, then $\mathcal{O}$ can be extended to one, $\mathcal{O}'$, by Lemma 3.1. Note that $\mathcal{O}'$ has the same rank as $\mathcal{O}$, and so in particular, $\mathcal{O}'$ is a line bundle.

To proceed, we will use the following fact, which we will leave unproved, but for a proof refer the reader to page 2 of [7].

Proposition 3.1. *An invertible sheaf of negative degree has no non-zero sections. An invertible sheaf of degree 0 has no non-zero sections unless it is the trivial sheaf, in which case it has a one-dimensional family of sections.*

Thus we have that $\mathcal{O}'$ must be of the form $\mathcal{O}(n)$ for some $n > 0$. This is because $\mathcal{O}'$ is a proper extension and thus has nontrivial global sections, implying that it cannot be isomorphic to either $\mathcal{O}$ or $\mathcal{O}(-n)$ for any $n \in \mathbb{N}$. Then since $\mathcal{O}'$ has positive degree, after one negative Serre twist (i.e. tensoring with $\mathcal{O}(-1)$), $\mathcal{O}'(-1)$ has nontrivial global sections. This is impossible, however, because $\mathcal{O}'$ is a sub-bundle of $V(-n)$ is nontrivial, and one negative Serre-twist of $V(-n)$ renders it without global sections.

So we have that $\mathcal{O}$ is indeed a sub-bundle of $V(-n)$ such that $V'(-n) := V(-n)/\mathcal{O}$ is also a vector bundle over $\mathbb{P}^1$. This gives us a short exact sequence of vector bundles

$$0 \to \mathcal{O} \to V(-n) \to V'(-n) \to 0, \tag{5}$$

giving the long exact sequence of cohomology groups

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}) \to H^0(\mathbb{P}^1, V(-n)) \to H^0(\mathbb{P}^1, V'(-n)) \to H^1(\mathbb{P}^1, \mathcal{O}) \to \cdots.$$

Note that by Proposition 3.18 in [5], $H^1(\mathbb{P}^1, \mathcal{O}) = 0$ and so

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}) \to H^0(\mathbb{P}^1, V(-n)) \to H^0(\mathbb{P}^1, V'(-n)) \to 0$$

is exact, giving that $H^0(\mathbb{P}^1, V'(-n)) \cong k^r/k \cong k^{r-1}$, because $H^0(\mathbb{P}^1, \mathcal{O}) \cong k$ and $H^0(\mathbb{P}^1, V(-n)) \cong k^r$. Note that if we twist (5) by $\mathcal{O}(-1)$ we get that $V'(-n-1)$ has no global sections, and furthermore, for all $\ell > 0$, $V'(-n-\ell)$ has no global sections. Thus the same n which was the maximal integer such that $V(-n)$ has nontrivial global sections and $V(-(n+1))$ does not is maximal with respect to the same property for V'.

We will show by induction on r that V can be decomposed as $V = \mathcal{O}(n)^{\oplus r} \oplus W$, where W is a vector bundle such that $W(-n)$ has no global sections. For our base case, let $r = 1$. We will consider what extension class $V(-n)$ belongs to—i.e., what class it corresponds to in $\mathrm{Ext}^1(V'(-n), \mathcal{O})$. Lemma 3.2 gives us that this corresponds to $H^1(\mathbb{P}^1, V'^*(n))$. As mentioned before, $V'^*(n)$ is dual to $V'(-n)$ and $\omega \cong \mathcal{O}(-2)$ for projective space, so we have by Serre Duality that

$$H^1(\mathbb{P}^1, V'^*(n)) \cong H^0(\mathbb{P}^1, V'(-n) \otimes \omega) \cong H^0(\mathbb{P}^1, V'(-n-2)) \cong 0$$

where the last isomorphism comes because $V'(-n-\ell)$ has no global sections for $\ell > 0$. Thus $V(-n)$ corresponds to the trivial extension of $V'(-n)$ by $\mathcal{O}$, meaning that $V(-n) \cong \mathcal{O} \oplus V'(-n)$. Note here that $V'(-n)$ cannot have any nontrivial global sections. Tensoring on the right by $\mathcal{O}(n)$, we get $V \cong \mathcal{O}(n) \oplus V'$.

Now let $r > 1$, and suppose that our statement holds for $r-1$. Note that the rank of V' is $r-1$, and so by the inductive hypothesis, we have the decomposition $V'(-n) \cong \mathcal{O}^{\oplus r-1} \oplus W(-n)$ where $W(-n)$ is a vector bundle with no global sections. Again we consider which extension $V(-n)$ might correspond to. Note that Ext is linear with respect to direct sums, and so

$$\mathrm{Ext}^1(V'(-n), \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{O}^{\oplus r-1} \oplus W(-n), \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{O}, \mathcal{O})^{\oplus r-1} \oplus \mathrm{Ext}^1(W(-n), \mathcal{O}).$$

Note that, again by Lemma 3.2 and Serre Duality,

$$\mathrm{Ext}^1(W(-n), \mathcal{O}) \cong H^1(\mathbb{P}^1, W^*(-n) \otimes \mathcal{O}) \cong H^1(\mathbb{P}^1, W^*(n)) \cong H^0(\mathbb{P}^1, W(-n-2)) \cong 0$$

and

$$\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}) \cong H^1(\mathbb{P}^1, \mathcal{O}) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2)) \cong 0.$$

Thus $\mathrm{Ext}^1(V'(-n), \mathcal{O}) \cong 0$, as desired, and we have that

$$0 \to \mathcal{O} \to V(-n) \to V'(-n) \to 0$$

splits, giving that $V(-n) \cong \mathcal{O} \oplus (\mathcal{O}^{\oplus r-1} \oplus W(-n)) \cong \mathcal{O}^{\oplus r} \oplus W(-n)$, where $W(-n)$ has no global sections. By twisting on the right by $\mathcal{O}(n)$, we get that

$$V \cong \mathcal{O}(n)^{\oplus r} \oplus W,$$

as desired.

Furthermore, W is a vector bundle with rank strictly smaller than the rank of V, and so we can decompose W as $W \cong \mathcal{O}(n')^{\oplus r'} \oplus W'$ for some $n', r' \in \mathbb{N}$ and vector bundle W'. Setting V' as W (not the same V' used before in this proof), this gives us a sequence of decompositions of the form

$$V^{(i)} \cong \mathcal{O}(n_i)^{\oplus r_i} \oplus W^{(i)}, \tag{6}$$

where $W^{(i)} = V^{(i+1)}$ and $V^{(0)} = V$. Since each resulting $W^{(i)}$ has rank strictly smaller than $V^{(i)}$, this process must terminate after finitely many steps and yield the decomposition:

$$V = \mathcal{O}(n_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}(n_q)^{\oplus r_q} \tag{7}$$

of V for some $q \in \mathbb{N}$.

We include a small remark that the explicit construction we have given for the decomposition of V yields that $n_1 > n_2 > \cdots > n_q$ because for each iteration, the vector bundle $W^{(i)}$ in the decomposition $V^{(i)} \cong \mathcal{O}(n_i)^{\oplus r_i} \oplus W^{(i)}$ has no global sections when twisted by $\mathcal{O}(-n_i)$. Therefore the n_{i+1} by which we take $W^{(i)}(-n_{i+1})$ in the next step of the decomposition of V must be strictly smaller than the previous n_i, due to the criteria which we imposed on each n_i, that it be the maximal integer such that $V(-n_i)$ has nontrivial global sections.

We will now show uniqueness via induction that the decomposition in (7) is uniquely determined by the n_i and r_i up to reordering. Suppose that

$$V \cong \mathcal{O}(a_1)^{\oplus s_1} \oplus \cdots \oplus \mathcal{O}(a_q)^{\oplus s_q}$$

such that $a_1 > a_2 > \cdots > a_q$, and consider the decomposition $V \cong \mathcal{O}(a_1)^{\oplus s_1} \oplus W$ where $W \cong \mathcal{O}(a_2)^{\oplus s_2} \oplus \cdots \oplus \mathcal{O}(a_q)^{\oplus s_q}$. If we twist V by $\mathcal{O}(-a_1)$ we get

$$V(-a_1) \cong \mathcal{O}^{\oplus s_1} \oplus W(-a_1).$$

Note that $\mathcal{O}$ has global sections (namely k), implying that $\mathcal{O}^{\oplus s_1}$ does as well, and finally that $V(-a_1)$ has nontrivial global sections. But if we twist by one more $\mathcal{O}(-1)$, by assumption $W(-a_1 - 1)$ has no global sections because $a_1 > a_j \; \forall \; j > 1$, and $\mathcal{O}(-1)^{\oplus s_1}$ has no global sections because $\mathcal{O}(-1)$ has none. So, in particular, $a_1 = n_1$ in our previous decomposition because a_1 is the maximal $n \in \mathbb{Z}$ by which we can twist V by $\mathcal{O}(-n)$ and have nontrivial global sections.

Now consider the global sections on $V(-a_1) \cong V(-n_1)$ which are $H^0(V(-n_1)) \cong k^r$ for some r, as given in our original decomposition above. Since $a_1 = n_1$, we have:

$$\mathcal{O}^{\oplus s_1} \oplus W(-a_1) \cong V(-n_1) \cong \mathcal{O}^{\oplus r_1} \oplus W'(-n_1) \tag{8}$$

where $W' \cong \mathcal{O}(n_2)^{\oplus r_2} \oplus \cdots \oplus \mathcal{O}(n_q)^{\oplus r_q}$. In particular, the isomorphism in (8) holds for global sections. Because $a_1 > a_j$ and $n_1 > n_j \; \forall \; j \neq 1$, both $W(-a_1)$ and $W'(-n_1)$ have no global sections. Thus

$$k^{s_1} \cong H^0(\mathcal{O}^{\oplus s_1}) \cong k^r \cong H^0(\mathcal{O}^{\oplus r_1}) \cong k^{r_1},$$

which gives $k^{s_1} \cong k^{r_1}$. Then k^{s_1} and k^{r_1} are isomorphic, finite-dimensional $k-$vector spaces and so $s_1 = r_1$. This is sufficient to prove uniqueness, because we can repeat the process just given on $V^{(2)} \cong \mathcal{O}(a_2)^{\oplus s_2} \oplus \cdots \oplus \mathcal{O}(a_q)^{\oplus s_q}$, etc. □

We remark here that the uniqueness at the end of the proof just given implies that no $\mathcal{O}(a)$ can be written as the sum of other line bundles $\mathcal{O}(a'_1) \oplus \cdots \oplus \mathcal{O}(a'_p)$ distinct from $\mathcal{O}(a)$. In particular, this is a proof of the fact mentioned before that the Picard group on $\mathbb{P}^1_k$ is freely generated by $\mathcal{O}(1)$ or $\mathcal{O}(-1)$ and thus isomorphic to $\mathbb{Z}$.

We will now conclude with the proofs of our lemmas which were essential to the proof of our main theorem. For the proof of Lemma 3.1, we're going to need an additional, small lemma as follows:

Lemma 3.3. *Let F be a coherent module on $\mathbb{P}^1_k$. Then either F has torsion or it is a locally free sheaf on $\mathbb{P}^1_k$. Every coherent sheaf F fits into a short exact sequence:*

$$0 \to F_t \to F \to F' \to 0 \tag{9}$$

where F_t is the torsion submodule and F' is locally free.

Proof. Suppose that F is torsion free. We will show that this implies that F is locally free. Note first that the stalks $\mathcal{O}_{\mathbb{P}^1_k,x}$ are discrete valuation rings for any $x \in X$ because $\mathbb{P}^1_k$ is a regular one-dimensional scheme. Also, by assumption F_x is free, say of rank r, by the classification of finitely-generated modules over a PID. Therefore on some affine, open neighborhood $U \ni x$, we have a surjection $\varphi : \mathcal{O}(U)^{\oplus r} \twoheadrightarrow F(U)$. We can restrict U to a smaller open set U' to avoid the supports of the kernel and cokernel of φ, and thus have an isomorphism $\varphi' : \mathcal{O}(U')^{\oplus r} \xrightarrow{\sim} F(U')$. Note that the choice of $x \in \mathbb{P}^1_k$ was arbitrary, so we have an open cover of $\mathbb{P}^1_k$ on which F is a free $\mathcal{O}_{\mathbb{P}^1_k}-$module, and F is locally free of rank r, as desired.

Then to fit F into the exact sequence in (9), we just let F_t be the torsion submodule, $F_t \to F$ be the inclusion map, and by the proof above, $F' := F/F_t$ is locally free (and thus a vector bundle, as coherence is preserved under quotients and both F and F_t are coherent sheaves). $\square$

We're now ready to prove Lemma 3.1.

Proof of Lemma 3.1. Suppose that V/W is not a vector bundle. Coherence is closed under quotients, and so it must be that V/W is not locally free. Thus by Lemma 3.3, V/W has torsion. We can extend W in V by the pullback of the torsion submodule T of V/W, which makes it fit into the exact sequence

$$0 \to W \to W' \to T \to 0.$$

Note that the (generic) rank of a torsion module is 0, and so we have immediately that the rank of W and the rank of W' is equal, and the same goes for the statement on generic rank of V/W with the rank of V/W'. Furthermore, V/W' is locally free (again, by Lemma 3.3), and so V/W' is a vector bundle on X. $\square$

And finally, the proof of Lemma 3.2 will finish our proof of Grothendieck's theorem.

Proof of Lemma 3.2. First note that $\mathrm{Ext}^i(V, -)$ is the i^{th} right derived functor of $\mathrm{Hom}(V, -)$, which we write as $\Gamma(\mathcal{H}om(V, -))$ (here, $\mathcal{H}om$ is the so-called 'sheafy hom'). Note that because V is a vector bundle, and thus locally free, $\mathcal{H}om(V, -)$ is exact. Our main step of this proof is to apply Grothendieck's composition of functors spectral sequence; in order to apply this we need to check that $\mathcal{H}om(V, -)$ maps injective coherent modules to injective coherent modules. To that end, suppose $\mathrm{Hom}(-, I)$ is exact (i.e., suppose that I is projective). Recall that $\mathcal{H}om(V, -) \cong V^* \otimes -$ (which is a generalization of $\mathrm{Hom}(V, -) \cong V^* \otimes -$ for modules over a ring). Thus $\mathrm{Hom}(-, \mathcal{H}om(V, I)) \cong \mathrm{Hom}(-, V^* \otimes I) \cong \mathrm{Hom}(- \otimes V, I)$ is exact, and we can apply Grothendieck's composition of functors spectral sequence. This gives

$$H^p(\mathcal{E}xt^q(V, F) \Rightarrow \mathrm{Ext}^{p+q}(V, F) \tag{10}$$

for any coherent F, where $\mathcal{E}xt^q$ is the i^{th} right-derived functor of $\mathcal{H}om$. But $\mathcal{E}xt^q(V, F) = 0$ if $q > 0$ because $\mathcal{H}om(V, -)$ is exact, and so from (10) we get the isomorphism

$$H^p(V^* \otimes F) \cong H^p(\mathcal{H}om(V, F)) \cong H^p(\mathcal{E}xt^0(V, F)) \cong \mathrm{Ext}^p(V, F)$$

for any coherent F, as desired. $\square$

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